**L’Hopital’s Rule & Indeterminate Forms**

**DEFINITION – Indeterminate Form**

A limit which, through direct substitution evaluates to

The above is an *indeterminate form*. We say this because we actually don’t know what’s going on in this equation.

This means the limit **does not exist** because it is an *undefined form*. We say, to symbolize the growth direction and magnitude of *x*, that this limit’s value is ∞.

*Note that* ∞ symbolizes **infinite growth***, not a particular value.*

∞/∞ forms

Example: (both of these go to infinity as their values get bigger)

0 times ∞ forms

Example: Lim x->0 ((x^2)cot(x^2)) (the cotangent function is not defined at 0, but is approaching infinity so we call it the infinity and the x^2 is defined and 0 at x = 0)

∞ - ∞ forms

Example: Lim x->0+(cot x – 1/x) (both approach infinity as x gets smaller)

0^0 forms

Example: lim x->0+ (x^x) (both grow smaller as x gets smaller)

∞^0 forms

Example: Lim x->2+ (4/(x^2 – 4))^(x-2) (base grows bigger, exponent grows smaller)

1^∞ forms

Example: Lim x->∞ (1 + 1/x)^(x) (base approaches 1, exponent approaching infinity)

**DEFINITION – L’Hopital’s Rule**

This rule gives us a way to deal with (and potentially get the value of) an indeterminate form. Suppose that:

Lim x->a f(x) = 0

lim x->a g(x) = 0

lim x->a f(x)/g(x) -> … -> DS 0/0 OR ∞/∞

* lim x->a f(x)/g(x) = lim x->a f’(x)/g’(x)

Note that there are technically special cases for this being a one sided limit or an infinite limit, but they are equivalent, just with a substituted for a-, a+, a ∞ or -∞.

Strategies

* Indeterminate multiplications can be made fractions
* Subtractions of infinities can be written under a common denominator.
* For indeterminate powers (0^∞, ∞^0, and 1^∞)

Examples of L’Hopital’s Rule

So now we return to our original equation from the

1. Lim x->1+ (ln x/(x-1)) ->DS 0/0 (see above)

= Lim x->1+ ((1/x)/1) *(By L’Hopital’s Rule)*

= Lim x->1+ ((1/x))

= 1

Note that this means that both have the same *growth behavior* as x->1+ because their ration as x->1+ is 1.

1. Lim x->∞ (e^x)/(x^2) ->DS ∞/∞

= Lim x->∞ (e^x)/(2x) ->DS ∞/∞ *(Via L’Hopital’s Rule)*

= Lim x->∞ (e^x)/(2) ->DS ∞/2 -> ∞

**Theorem:** lim x->∞ (e^x)/(x^n) = ∞ (this will be obvious by the end of the course)

1. Lim x->0 (x^2cot(x^2))  
   Note that we can rewrite this as a product  
   = Lim x->0 (x^2)/(1/cot(x^2)) *(Note that we can take another path here, but it doesn’t clean up as nicely)*  
   = Lim x->0 (x^2)/(tan(x^2))  
   = Lim x-> 0 (2x)/(2xsec^2(x^2))  
   = Lim x->0 1/sec^2(x^2)  
   = Lim x->0 cos^2(x^2)  
   ->DS 1
2. Lim x->0 (cot x – 1/x) ->DS ∞ - ∞  
   = Lim x->0 ((cos x/sin x) – 1/x)  
   = Lim x->0 ((xcosx) – sin x)/xsinx)  
   = Lim x->0 ((cos x – xsin x – cos x)/(sin x + xcos x)) *(L’Hopital)*  
   = Lim x->0 (-xsin x)/(sin x + xcos x)  
   = Lim x->0 (-sin x – xcos x)/(cos x + cosx – xsin x)  
   ->DS 0
3. Lim x->∞ (1 + 1/x)^x  
   Let y = (1 + 1/x)^x, then lny = ln((1 + 1/x)^x)   
    = xln(1 + 1/x)

Then,  
Lim x->∞ ln y = Lim x->∞ xln(1 + 1/x)  
Note that now we have the limit of ∞ times ∞  
So we make a fraction.  
We turn this into:  
Lim x->∞ ln(1 + 1/x)/(1/x) ->DS 0/0  
= Lim x->∞ ((1/(1 + 1/x)) + (-1/x^2))/(-1/x^2) *(By L’Hopital’s Rule)*= Lim x->∞ (1/(1 + 1/x)) ->DS 1  
  
**BUT this is the limit of ln y (y being our answer)**So we get  
Lim x->∞ (y) = Lim x->∞ e^(ln y)  
= e^(Lim x->∞ (ln y)) *(because e^x is continuous)*= e^1 = e

*Further examples can be found in the book in examples 1-9 p 303-307. We have done example 1.*

Be careful!

Lim x->1+ (x^2)/(ln x) ->DS 1/0 -> ∞f

But what if we didn’t realize this **is not** indeterminate and used L’Hopital’s Rule?

= Lim x->1+ (2x)/(1/x) ->DS 2/1 = 2

**We get the wrong answer!**

**Make SURE that your limit follows the requirements of L’Hopital’s Rule.**

***Exercises can be found on the course website. For this section they are:***

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